# ANALYSIS OF RECURRENCE PLOT VS AUTOCORRELATION BASED FREQUENCY ESTIMATION METHODS

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**Keywords:** Fundamental frequency estimation, Recurrence plot analysis, Autocorrelation function, State space representation.

**Abstract:** *Estimation of fundamental frequency is an area of interest in communications and speech processing, as well as in the investigation of various physical phenomena. We make in this paper a comparative analysis between the basic time domain approaches and those based on the reconstructed state space representation. We present the philosophies of these two directions and we propose a fundamental frequency estimation method based on quantification of diagonal lines in the recurrence matrix. We show that this method is in fact an extension of the difference function method. We also comparatively show results obtained by using these methods for different signals and for different SNRs (signal-to-noise ratios).*

## INTRODUCTION

Periodic signals are of great importance in signal analysis, as they are the very basis on which the core of signal analysis - that is the Fourier transform - has developped. A periodic signal is characterized by the fact that is repeats itself identically after a certain time period $T_0$, which is the inverse of the fundamental frequecy $f_0$. In practice, periodic signals don't exist - they are just a mathematical idealization. Noise, as well as the complexity of the interactions that real-life systems are subject to, they all lead to a periodicity that is only approximate. Even more, when the perturbations that affect the system are very strong, this periodicity becomes hardly or not at all noticeable. The domain of frequency estimation has as its purpose the determination of $f_0$ for a signal that is quasi-periodic. (As an example, we can consider a periodic signal affected by noise in a communication system; or sounds corresponding to vowels - knowing that they generate quasi-periodic signals.) Estimating $f_0$ is a challenging task, considering that for a signal that isn't perfectly periodic the notion of fundamental frequency isn't even defined. Despite this mathematical fact, the analyzers that a human being possesses are able to give a (mostly qualitative than accurately quantitative, it's true) image regarding this fundamental frequency.

Computational methods for the estimation of fundamental frequency are split into two classes: frequency domain methods and time domain methods. Working in the frequency domain seems to be a natural choice, considering that it's the frequency that one wants to determine. On the other hand, time domain methods target not the estimation of the frequency, but the estimation of the signal's period. Most of these methods have as their starting point the autocorrelation function (ACF). It measures the self-similarities that exist in a signal. A periodic signal generates maxima in the ACF at the time instants where the signal repeats itself. Based on a similar principle, good results have been obtained by using differences instead of the multiplications involved by the ACF. [1]

* This article was produced under the project "Supporting young Ph.D students with frequency by providing doctoral fellowships", co-financed from the EUROPEAN SOCIAL FUND through the Sectoral Operational Program Development of Human Resources.

An approach that is closely related to the time domain approach, but has a different working philosophy, is reconstructing the state space trajectory of the signal [2] [3]. In state space representation, a periodic signal becomes a closed trajectory. Returning to a previously visited point marks the end of a cycle that corresponds to a period of the signal. This manner of seeing thing handles naturally the possibility that the signal is not perfectly periodic. This is done by using the concept of recurrence that is defined as returning close to a previously visited point, and not exactly in that point. The parallelism between the concept of frequency and that of recurrence suggests using the latter to estimate the former.

In this paper we study a method to estimate the period of a signal starting from the bidimensional representation of recurrences that is known as recurrence plot. [4]

The paper continues with Section 1, that contains a brief description of the frequency estimation methods that we studied. In Section 2 results of applying the described methods in order to estimate frequency for some (noiseless, as well as noisy) signals are shown. Section 3 discusses the presented results, comparing the methods by various criteria. The paper is closed by some conclusions.

## 1. FREQUENCY ESTIMATION

As mentioned previously, what the time domain frequency estimation methods actually estimate is the signal's period. These methods start from the idea of signal self-similarity that is mathematically quantified by the function of autocorrelation. Similar, but much less computationally demanding results can be obtained by using the difference function (as will be shown). When a signal is shifted from itself with a period, the corresponding samples have close values. So, by multiplying and then summing them, big values are obtained (and correspondingly, by subtracting them and then summing their modulus small values are obtained). Shifting the signal against itself is also the starting point for analyzing the recurrences of state space trajectory. The next step is, as will be seen, computing the distances between all the pairs of two points from the trajectory. The distances that are below certain fixed threshold represent recurrences, and they can be quantified in order to estimate the period of the analyzed signal.

## A. AUTOCORRELATION FUNCTION

For a signal $s$ having $N$ samples, the ACF is computed as:

$$ACF(\tau) = \frac{1}{N-\tau}\sum_{n=1}^{N} s[n]s[n+\tau]\,. \tag{1}$$

When the delay $\tau$ is very close to the fundamental period $T_0$ of the signal, the ACF has a local maximum, as shown in Fig. 1.

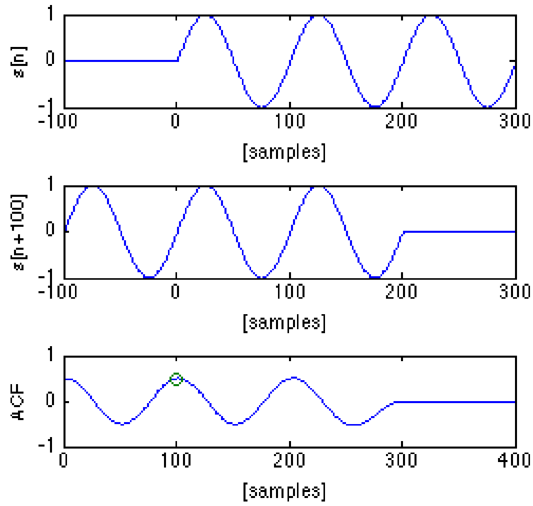


**Fig. 1:** Illustration of the working principle of the ACF.

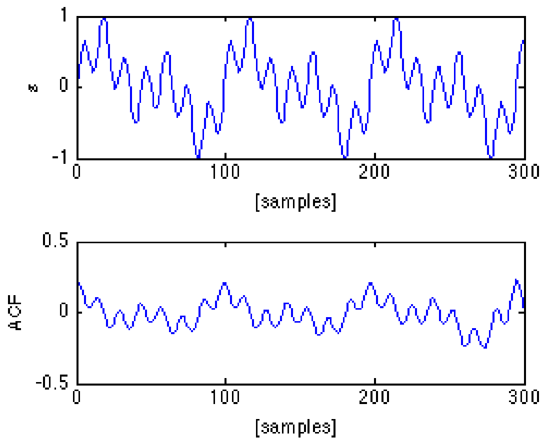


**Fig. 2:** The ACF for a three-component periodic signal.

ISSN – 1453 – 1119

The ACF is very noise-resistant, but has the disadvantage that detecting the fundamental becomes a difficult task when the periodic signal contains more than one spectral component. This is shown in Fig. 2.

By replacing the multiplication in Eq. (1) with a subtraction, the AMDF (averaged magnitude difference function) is obtained:

$$AMDF(\tau) = \frac{1}{N-\tau}\sum_{n=1}^{N} |\, s[n] - s[n+\tau]\,| \,. \qquad (2)$$

A variant [1] of it is the ASDF (averaged squared difference function):

$$ASDF(\tau) = \frac{1}{N-\tau}\sum_{n=1}^{N} (s[n] - s[n+\tau])^2 \,. \qquad (3)$$

(Observation: The normalization used in these definitions is not the only one that is possible.)

## B. RECURRENCE MATRIX

Recurrence analysis starts with representing the signal as a trajectory in state space. Periodic signals are closed curves in this representation. When signals aren't perfectly periodic (as the result of their affecting by noise, for example), these curves aren't perfectly closed curves any more, but they form a *cloud* around the original trajectory.

A signal can be represented as a trajectory in state space by using the method of delays [5]. Fig. 3 shows an example for a sinusoidal signal.

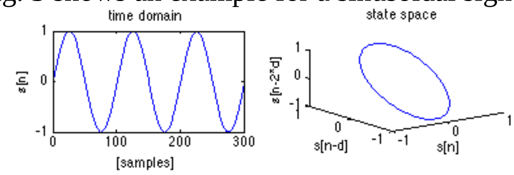


**Fig. 3:** State space trajectory representation for a sinusoidal signal. The reconstruction of the trajectory is done using a dimesion of 3 and a delay of 15 samples.

Adding noise to that sinusoidal signal leads to a trajectory as that shown in Fig. 4.

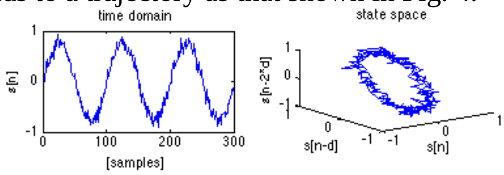


**Fig. 4:** State space representation for a sinusoidal signal corrupted by noise at 15dB SNR. Reconstruction is done using a dimension of 3 and a delay of 15 samples.

For multi-component periodic signals, the trajectory becomes a curve that is more complicated than an ellipse, but is still closed (and the effect of noise remains similar to what can be seen in Fig. 4).

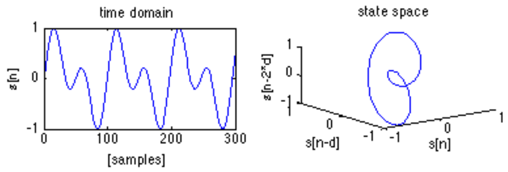


**Fig. 5:** State space representation for a two-component periodic signal, using three dimensions and a delay of 5.

We recall that representing the signal as a trajectory by the method of delays involves choosing first two parameters: *m* (the state space dimension) and *d* (the time delay between coordinates). Their choice is dicussed in [5].

By measuring the distances between every pair of two points from the state space trajectory and by marking with a dot the pairs for which this distance is below a certain threshold, one obtains the recurrence plot. In it, periodic signals, characterized by wide curves in state space, become diagonal lines paralell to the main diagonal, as Fig. 6 shows.

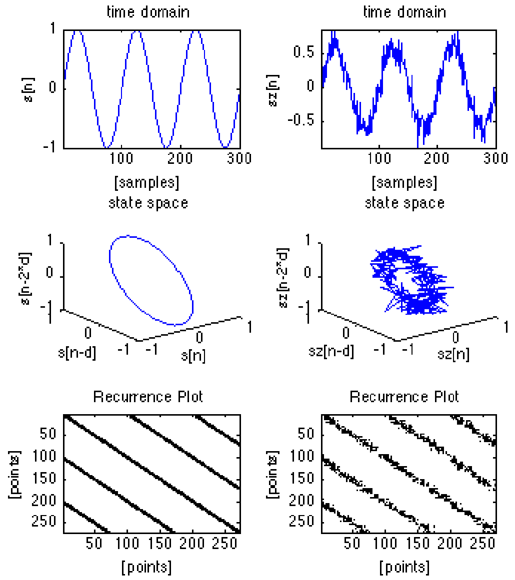


**Fig. 6:** Comparative illustration for three representations of a signal - time domain, state space trajectory, and recurrence plot. (left) noiseless signal. (right) signal at 10dB SNR.

Hence, the period of the signal can be estimated by estimating the distances between two such consecutive diagonal lines. A very intuitive method to perform this task consists in summing diagonal-wise the elements in the recurrence matrix. We recall that the recurrence plot is a graphical representation of the recurrence matrix that is defined as:

$$RM_{i,j} = H(\varepsilon - \| \overrightarrow{rec_i} - \overrightarrow{rec_j} \|), \qquad (4)$$

where *H* is the Heaviside step function, *ε* is the chosen recurrence threshold (which is, therefore, a parameter of the method), and $\overrightarrow{rec_i}$ designates the state vectors reconstructed by the method of delays from signal *s*, using parameters *m* and *d*, as follows:

$$\overrightarrow{rec_k} = (s[k], s[k+d], ..., s[k+(m-1)d]). \qquad (5)$$

Therefore, the (normalized) summing of the elements on the diagonals of the recurrence matrix (- we're referring only to the elements above the main diagonal, as this matrix is symmetrical -) translates into:

$$SD(\tau) = \frac{1}{N-\tau} \sum_{i=1}^{N-\tau} H(\varepsilon - \| \overrightarrow{rec_i} - \overrightarrow{rec_{i+\tau}} \|). \qquad (6)$$

Concerning the parameter *ε*, one must notice that interpreting the topological relation between two points on the trajectory as being a recurrence or not depends directly on this parameter. We want *ε* to be, on one hand, sufficiently big so as to cancel the effect of noise - hence, it should be at least equal to the largest width of the cloud of trajectories that noise generates around the original trajectory. In this way recurrences are caught in the recurrence matrix. A *ε* that is too big, on the other hand, leads to the lost of signification for a recurrence, as the recurrence ball obtained with a big *ε* contains a significant portion of the trajectory (if not all of it!).

## C. CONNECTION BETWEEN THE TWO METHODS

It is clear that the representation of a signal as a state space trajectory doesn't represent a transformation of the signal's domain. The domain is still the time, closely connected to the index of each point on the trajectory. This relation isn't, however, bi-univocal, but depends on the size of the analysis window, given by:

$$w = (m-1)d \qquad (7)$$

This *w* shows the distance in time between the most distant signal samples that contribute (as coordinates in state space) to the construction of a certain point on the trajectory. For complicated trajectories (that usually correspond to signals with a rich spectrum), the optimum choice of the reconstruction parameters *m* and *d* often leads to a big analysis window. The advantage of an optimum choice for *m* and *d* is a wide trajectory (without self-intersections), but it has the disadvantage that it increases the imprecision in associating points of the trajectory to a certain time instant - this time instant can be located anywhere on a temporal area that directly depends on the size of *w*.

In the particular case of reconstructing the trajectory with no embedding (that is, *m*=1 and *d*=1), we notice that *w*=0, which makes possible a bi-univocal correspondence between trajectory's points and the samples' time instants. We are speaking about a *trajectory* even in this uni-dimensional case, because all discussions so far stay true in this case, too. Even more, the recurrence plot contains in it many similarities with the one that correspond to the properly reconstructed trajectory. In the no embedding case, Eq. (6) becomes:

$$SD(\tau) = \frac{1}{N-\tau} \sum_{n=1}^{N-\tau} H(\varepsilon - | s[n] - s[n+\tau] |) \qquad (8)$$

We notice that Eq. (8) - that represents the sum of diagonals in the recurrence matrix - is similar to Eq. (2). SD can be obtained, in this case, by introducing in the AMDF nonlinearity made from a combination of the Heaviside function and the parameter *ε*. Hence, this method to estimate frequency from the state space trajectory can be seen as an extension (through *H*() and *ε*) and a generalization (through *m* and *d*) of the AMDF method.

## 2. RESULTS

Fig. 7 shows comparatively some results of applying the previously described methods on two types of signals - a mono-component one, and a signal with three spectral components. We must mention that for the results presented in this section we used no embedding (*m*=1 and *d*=1) and for $\varepsilon$ we used the mean distance between successive points on the trajectory.

Repeating the same experiments for noisy conditions, leads to the results shown in Fig. 8.

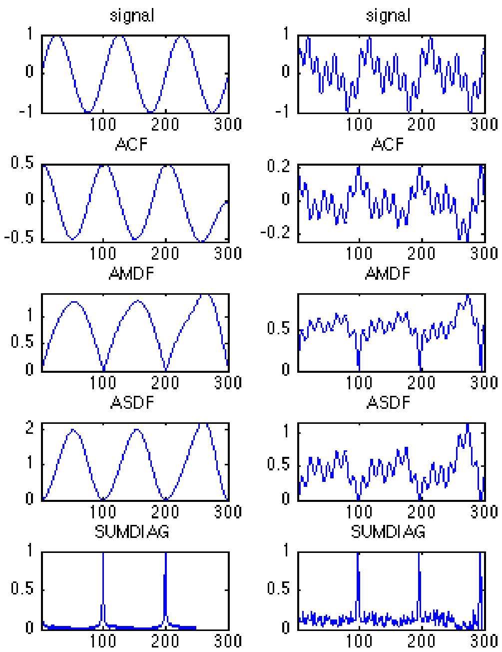


**Fig. 7:** Comparative results for the time domain (ACF, AMDF, ASDF) and the recurrence plot based methods. (left) a sinusoidal signal. (right) a three-component signal.

## 3. DISCUSSION

The results in Fig. 7 show that the ACF has significant maxima for the high frequency components in the signal, which makes fundamental frequency identification more difficult. The influence of these high frequency components is reduced if the difference functions are used instead. AMDF has also the advantage of a less expensive computation, as the multiplications in the ACF are replaced with subtractions, which are much faster to compute.

The curves obtained by summing the diagonals of the recurrence matrix perform a clear separation of the fundamental period, regardless of whether the signal has other components as well, or not. As expected (from the discussions in the preceding Section, regarding the choice of $\varepsilon$), the SD method's behavior is strongly dependent on the SNR. This is shown in Fig. 8. This behavior could be slightly improved at low SNRs by using the optimum parameters for embedding. Experiments (whose results aren't shown in this paper) show that minor improvements could also be obtained by filtering the recurrence plot [3] before computing SD. We must, however, make the observation that in the no embedding case we used, SD is (similarly to AMDF) efficiently computable.

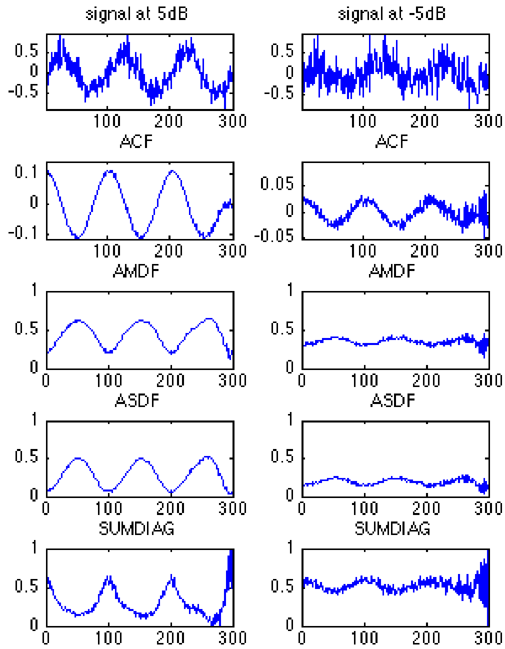


**Fig. 8:** Comparative results obtained for a sinusoidal signal, for two SNRs - (left) 5dB; (right) -5dB.

All the remarks made so far would suggest combining AMDF and SD - by successively applying them, starting with the analyzed signal. The resulting method would benefit, on one hand, on the noise robustness of the autocorrelation function (that its derivates also possess) and, on the other hand, on the good fundamental period separation offered by the SD method.

We must point out that we haven't tackled here also the problem of post-processing the curves that we obtained, in order to produce an estimation of the fundamental period. This can be done, for example, by averaging the distances between successive maxima (or minima, in the case of difference functions). Furthermore, for increasing the noise robustness certain filtering of the obtained maxima (or minima) may be performed before their averaging.

## 4. CONCLUSION

In this paper we presented a method to estimate the fundamental period of periodic signals, based on the quantification of the recurrence plot. We have shown that there are similarities between this method and a time domain method, derived from the autocorrelation function. We also illustrated comparatively results obtained with these two types of methods, pointing out the advantages and the disadvantages of each of them.

Finally, we proposed a combination between the analyzed methods, in order to increase the performances (in terms of noise robustness and fundamental frequency separation).

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